

## A Proof of Convergence of the Horn–Schunck Optical Flow Algorithm in Arbitrary Dimension\*

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**Abstract.** The Horn–Schunck (HS) method, which amounts to the Jacobi iterative scheme in the interior of the image, was one of the first optical flow algorithms. In this paper, we prove the convergence of the HS method whenever the problem is well-posed. Our result is shown in the framework of a generalization of the HS method in dimension  $n \geq 1$ , with a broad definition of the discrete Laplacian. In this context, the condition for the convergence is that the intensity gradients not all be contained in the same hyperplane. Two other works ([A. Mitiche and A. Mansouri, *IEEE Trans. Image Process.*, 13 (2004), pp. 848–852] and [Y. Kameda, A. Imiya, and N. Ohnishi, *A convergence proof for the Horn-Schunck optical-flow computation scheme using neighborhood decomposition*, in *Combinatorial Image Analysis*, Springer, Berlin, 2008, pp. 262–273]) claimed to solve this problem in the case  $n = 2$ , but it appears that both of these proofs are erroneous. Moreover, we explain why some standard results about the convergence of the Jacobi method do not apply for the HS problem, unless  $n = 1$ . It is also shown that the convergence of the HS scheme implies the convergence of the Gauss–Seidel and successive overrelaxation schemes for the HS problem.

**Key words.** optical flow, Horn–Schunck algorithm, Jacobi iterations

**AMS subject classifications.** 68U10, 11D04, 05C50, 65F10, 65N22, 65F35

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**1. Introduction.** Optical flow refers to the distribution of apparent movement of intensity patterns in an image caused by the relative motion between an observer and the scene. The Horn–Schunck (HS) method was one of the first optical flow algorithms used to determine a displacement field from several successive images [10]. The original HS method is based on a global approach and introduces a quadratic prior term of smoothness in the classical equation of the optical flow. This algorithm is especially adapted to speckled or diffuse images like

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those encountered in several modalities where a displacement field without discontinuity or significantly high gradients is expected [24, 16, 14]. Thus, the HS method and its derived forms remain of high interest in some areas of motion imaging. Other complex and very proficient estimators for optical flow, however, now exist in the context of natural scenes [2, 3] to take into account discontinuities at object edges.

Based on a discretization of the differential operators appearing in the HS optical flow formulation, the HS method results in a linear system that can be solved with direct or iterative methods. In comparison to direct methods, the iterative solvers have the advantage of needing lower computational data storage and to be easily programmable. It is well known that the matrix involved in the HS linear system is symmetric positive definite, as a consequence of the V-ellipticity of the HS functional [18]. This ensures, for example, the efficiency of the direct Cholesky decomposition and of the iterative Gauss–Seidel or successive overrelaxation (SOR) solvers. However, the positive definiteness does not permit one to conclude about the method proposed in Horn and Schunck’s initial paper, which is an iterative  $2 \times 2$  blockwise solver [10] and corresponds to the Jacobi solver for the interior points of the image only. In fact, it is shown in this work that the positive definiteness is implied by the convergence of the HS scheme. The Gauss–Seidel and SOR solvers are known to converge at least twice as fast as the Jacobi solver [4, Theorem 5.3-4]. These iterative solvers can be made parallelizable using, for instance, a special red-black reordering of the unknowns in the linear system [6, 25]. The Jacobi iterative solver, however, has the advantage of being directly parallelizable since it does not use values computed in the current iteration step [20, 21].

One known general result about the convergence of the Jacobi method concerns strictly (block) diagonally dominant matrices, which is not the case here. Another result concerns (block) irreducible and weakly dominant matrices, an assumption which is not satisfied for images of dimension greater than 1 under the appropriate Neumann boundary conditions. Whether the iterative method for the HS linear system with the Neumann boundary conditions converges still remains unsolved. Indeed, the paper of Horn and Schunck did not include a proof of convergence [10]. Two proofs of convergence have been published since then, in [17] and [13], both for 2-dimensional images. However, as far as we can tell, these two proofs are erroneous. There is also a short argument in [23, p. 249] based on diagonally dominant matrices (without blocks), for the convergence of the pointwise Jacobi method, that is erroneous.

Under a general perspective, there are three main points in an optical flow algorithm: (1) the formulation of the continuous energy (functional) to be minimized; (2) the discretization scheme; and (3) the solver used to minimize the energy. The scope of this work is to present a proof that the HS iterative solver (and hence the Gauss–Seidel and SOR solvers) converges for the original quadratic HS functional under a generic discretization scheme adopted in this paper.

In section 2, we state a generalization of the HS method in dimension  $n$ . In section 3, we explain why the previous proofs are erroneous and cannot be fixed. In section 4, we define some hypotheses about the discrete Laplacian, propose a necessary and sufficient condition for the linear system of Horn and Schunck to be invertible, and state our convergence result. The proof is presented in section 5. In section 6, we define a general way of calculating a discrete Laplacian in dimension  $n$ . In section 7, we show that our general discrete Laplacian satisfies the hypotheses imposed to get the convergence result. In Appendix A, the HS iterative

scheme is derived in detail from the discretization of the HS problem. In Appendix B, it is explained why the coefficient matrix of the HS scheme is not strictly (block) diagonally dominant matrices, nor (block) irreducible and weakly dominant matrices under the appropriate Neumann boundary conditions for images of dimension greater than 1. A result is shown in Appendix C that implies the convergence of the Gauss–Seidel and SOR iterative schemes whenever the Jacobi method converges, under appropriate conditions. This result also implies that the Gauss–Seidel and SOR methods converge for the HS problem, as a consequence of the convergence of the HS iterative scheme. In Appendix D, details are given to explain why the proofs of [17, 13] are erroneous.

**2. Statement of the problem.** The optical flow problem is usually applied to 2-dimensional images of a moving scene [10]. Optical flow has also been used to analyze motion in one, three, or four dimensions [22, 9, 5]. In this work, we investigate the convergence of the HS optical flow problem in the generalized case of dimension  $n \geq 1$ . We thus consider an orthotope  $V \subset \mathbb{R}^n$ , i.e., a parallelotope whose edges are all mutually perpendicular (a segment if  $n = 1$ , a rectangle if  $n = 2$ , or a cuboid if  $n = 3$ ). In the optical flow problem, each element of  $V$  generally corresponds to an intensity or brightness that varies over time. Given the intensity field over two or more successive instants, the aim of the HS method is to determine the corresponding displacement field. As we propose here a proof of convergence in the context of  $n$ -dimensional arrays, we first state an  $n$ -dimensional generalization of the HS method (for the classical 2-dimensional formulation, we refer the reader to [10]).

Let  $I$  denote the intensity field on  $V$ ,  $I_t$  its derivative with respect to time  $t$ ,  $\nabla I$  its gradient with respect to position, and  $\mathbf{u}$  the displacement field. We start from the well-known optical flow identity:

$$(2.1) \quad \nabla I \cdot \mathbf{u} + I_t = 0,$$

which means that a given (apparently moving) point of  $V$  keeps its initial intensity during its displacement. Then, a regularization method is employed to impose low spatial variations in the displacement field. By definition, the HS method consists in minimizing the unconstrained functional:

$$(2.2) \quad J(\mathbf{u}) = \int_V \{(\nabla I \cdot \mathbf{u} + I_t)^2 + \mu \|\nabla \mathbf{u}\|^2\} dV,$$

where  $\mu > 0$  is a positive real number and  $\|\cdot\|$ , in the entire paper, represents the Euclidean norm. The Euler–Lagrange equation corresponding to this minimization problem reads as follows:

$$(2.3) \quad \mu \Delta \mathbf{u} = (\nabla I \cdot \mathbf{u} + I_t) \nabla I = [\nabla I \nabla I^T] \mathbf{u} + I_t \nabla I \quad \text{on } V,$$

$$(2.4) \quad \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0 \quad \text{on } \partial V.$$

Here,  $\frac{\partial}{\partial \mathbf{n}}$  is the differentiation operator in the direction of the normal  $\mathbf{n}$  to the boundary  $\partial V$ , and the superscript  $T$  denotes transposition of matrices. The displacements without the superscript  $T$  are considered to be column vectors (in  $\mathbb{R}^n$ ). Note that the Neumann boundary conditions (2.4) arise naturally from the unconstrained minimization problem (2.2).

Now, we will discretize the expression of (2.3) on a lattice  $\Lambda$  covering the orthotope  $V$ . The restriction of the intensity field on the lattice  $\Lambda$  can be viewed as a (discretized) image. In what follows, we assume that there are  $N \geq 2$  elements in the lattice  $\Lambda$ . Then, a discretized displacement field is of the form  $\mathbf{u} = (\mathbf{u}_i)_{i \in \Lambda}$ , where  $\mathbf{u}_i = (u_{i1}, \dots, u_{in})^T$  denotes the displacement vector at the point  $i$ . In the following, we will say that a displacement field  $\mathbf{u}$  is uniform if all the displacement vectors  $\mathbf{u}_i$  are identical. Now, (2.3) can be written for  $i \in \Lambda$  as

$$(2.5) \quad \mu \Delta(\mathbf{u})_i = [\nabla I \nabla I^T]_i \mathbf{u}_i + I_{t,i} \nabla I_i,$$

where  $I_{t,i}$  denotes the partial derivative of the intensity  $I$  with respect to  $t$  evaluated at the point  $i$ . In (2.5),  $\Delta(\mathbf{u})_i$  is a discretized Laplacian that depends linearly on the vectors  $\mathbf{u}_i$  for  $i \in \Lambda$ . Hence, the consideration of (2.5) for  $i \in \Lambda$  yields a linear system of  $nN$  equations and  $nN$  unknowns, where  $N$  is the number of points in  $\Lambda$ . The discretization of the Laplacian can classically be written as

$$(2.6) \quad \Delta(\mathbf{u})_i = \kappa \{M(\mathbf{u})_i - \mathbf{u}_i\},$$

where  $\kappa > 0$  is a positive real number and  $M$  a linear transformation (on the vector space of displacement fields) that returns for each point an average of the displacement field over its neighbors:

$$(2.7) \quad M(\mathbf{u})_i = \sum_{j \in \Lambda} \lambda_{ij} \mathbf{u}_j,$$

where  $\lambda_{ij}$ , for  $i, j \in \Lambda$ , are nonnegative real numbers. We will adopt in section 6 a general expression of this operator. In the following, we denote for notational convenience

$$(2.8) \quad \alpha = \mu \kappa,$$

where  $\mu$  is the regularization weight of (2.2), so that the coefficient  $\alpha$  is a positive real number. In order to solve the linear system (2.5), Horn and Schunck [10] proposed an iterative method that is assumed to converge to the solution. Let  $P$  be the linear transformation (on the vector space of displacement fields) defined by  $P(\mathbf{u})_i = P_i \mathbf{u}_i$  for  $i \in \Lambda$ , where  $P_i = \mathcal{I}_n - \frac{[\nabla I \nabla I^T]_i}{\alpha + \|\nabla I_i\|^2}$  and  $\mathcal{I}_n$  is the  $n \times n$  identity matrix. Let  $\mathbf{d}$  be the displacement field defined by  $\mathbf{d}_i = -\frac{I_{t,i} \nabla I_i}{\alpha + \|\nabla I_i\|^2}$  for  $i \in \Lambda$ . Then, the HS iterative scheme is expressed as follows:

$$(2.9) \quad \mathbf{u}^{k+1} = P M(\mathbf{u}^k) + \mathbf{d}.$$

See Appendix A for a derivation of (2.9). Also, it is shown in Appendix B that the HS iterative scheme of (2.9) amounts to the Jacobi iterative scheme in the interior of the orthotope  $V$ , but never at its boundary points. Moreover, in that appendix, we explain why standard results (based on block diagonally dominant matrices) on the convergence of the Jacobi iterative scheme do not apply in this context, due to the natural Neumann boundary conditions (2.4). We also explain why the short argument of [23, p. 249] based on diagonally dominant matrices (without blocks) for the pointwise Jacobi method is erroneous for the HS problem.

In Appendix C, it is shown that the convergence of the Gauss–Seidel and SOR iterative schemes is implied by the convergence of the Jacobi method, under appropriate conditions, based on a result about symmetric positive definite matrices. In particular, this result implies that the Gauss–Seidel and SOR methods converge for the HS problem, as a consequence of the convergence of the HS iterative scheme.

It would be straightforward to prove the convergence of the Jacobi solver in the presence of Dirichlet boundary conditions since the matrix would be block irreducible and weakly block diagonally dominant in that case. However, we recall that the Neumann conditions are intrinsically related to the minimization of the cost function (2.2).

**3. Previous proofs.** As stated in the introduction, the two existing proofs of convergence of the Jacobi solver in the context of the HS problem ([17] and [13]) appear to be erroneous. Let us now see in detail where the errors occurred and why we think that they cannot be fixed.

In [17], the cornerstone of the proof of convergence of the Jacobi method for solving the HS linear system relies on [17, eq. (16)], which states that the function defined by the matrix “ $P$ ” of [17, eq. (9)] (not to be confused with the linear transformation  $P$  of (2.9) of the present paper) is contracting for the norm defined by [17, eq. (10)], for any image. However, it appears that the only case for which this can be true is if the image is uniform, as explained in detail in Appendix D.

In [13, eq. (20)], we notice that no condition is given for the convergence of the HS iterations. However, in view of Theorem 4.1 below, that assertion is false (a condition on the image gradients is needed to make the HS problem well-posed). Thus, the proof in [13] must be erroneous. In Appendix D, we give further details on intermediate statements that are false in [13].

Finally, in [23, p. 249], the special case of the HS problem amounts to  $\Psi'(s^2) = 1$  (see also [23, p. 247, second column]). In that case, the iterations of [23, eqs. (14) and (15)] amount to the Jacobi iterative scheme for the system (2.5), but without considering  $n \times n$  blocks. It is asserted that “If the discrete image gradient does not vanish at one point, the system matrix of these equations is irreducibly diagonally dominant. This guarantees the existence of a unique solution of the linear system and global convergence of the Jacobi iterations [26]”. But, as shown in Appendix B, the coefficient matrix of the system is not even diagonally dominant, except in a special case. Thus, that argument is also erroneous.

**4. Statement of the main result.** First, the operator  $M$  of (2.6) and (2.9) comes from the Laplacian discretization and returns, for each point, an average of the displacement field over its neighbors as in (2.7). We will have to define several hypotheses about  $M$ . In what follows, we will assume the following:

(H1) For all points  $i$  and  $j$  of  $\Lambda$ ,  $\lambda_{ij} = \lambda_{ji}$ .

(H2) At every point  $i$  of  $\Lambda$ ,  $\sum_{j \in \Lambda} \lambda_{ij} = 1$ .

Intuitively, (H1) comes from the isotropy property of the smooth Laplacian, and (H2) is necessary in order to have a null Laplacian when the displacement field is uniform. As we will see in section 7, these hypotheses are verified with the general discretization scheme of section 6. In order to state the last hypothesis, we have to define the graph  $G$  by its set of vertices  $V(G) = \Lambda$  and its set of edges  $E(G) = \{(i, j) \in \Lambda^2 : \lambda_{ij} \neq 0\}$ . If  $(i, j) \in E(G)$ , we write  $i \sim_G j$ .

We assume that the lattice  $\Lambda$  is of the form  $\{(i_1, i_2, \dots, i_n) : i_\ell \text{ is an integer ranging from } 0 \text{ to } N_\ell - 1 \text{ for } 1 \leq \ell \leq n\}$ , where  $N_\ell \geq 1$  for  $1 \leq \ell \leq n$ . Thus, the number of points in  $\Lambda$  is equal to  $N = \prod_{\ell=1}^n N_\ell$ . From (H1), this graph is undirected (i.e.,  $i \sim_G j$  if  $j \sim_G i$ ). Let us now recall that an undirected graph  $G$  is connected if, for any two vertices  $i$  and  $j$  of  $G$ , there exists a path from  $i$  to  $j$  in  $G$ . We can now state the last hypothesis:

(H3) The graph  $G$  is connected.

We will see in section 7 that (H3) is also true with the general discretization scheme of section 6. Actually, this is an immediate consequence of the fact that the closest neighbors of a point are taken into account in the average calculation at this point. In the following, we will call *rank of  $(\nabla I_i)$*  the dimension of the subspace of  $\mathbb{R}^n$  that is spanned by the vectors  $\nabla I_i$ ,  $i \in \Lambda$ .

**Theorem 4.1.** *Under hypotheses (H1), (H2), and (H3), the following hold:*

- *If the rank of  $(\nabla I_i)$  is  $n$ , the linear system (2.5) has a unique solution and the iterations (2.9) converge to this solution.*
- *If the rank of  $(\nabla I_i)$  is not  $n$ , the problem is ill-posed; i.e., the linear system (2.5) does not have a unique solution.*

Let us notice that the rank of  $(\nabla I_i)$  is different from  $n$  if and only if the intensity gradients are all contained in the same hyperplane. In that case, the image is invariant along the direction orthogonal to this hyperplane. The fact that this condition makes the problem ill-posed is not surprising, as it is clear that a displacement along this particular direction cannot be detected by studying the variations of intensity over time.

**5. Proof of the main result.** The linear transformation  $M$  and the coefficients  $\lambda_{ij}$  are defined in (2.7). The linear transformation  $P$  and the matrix  $P_i$  are defined in section 2 before (2.9). We define the norm of a displacement field  $\mathbf{u}$  by  $\|\mathbf{u}\| = (\sum_{i \in \Lambda} \|\mathbf{u}_i\|^2)^{1/2}$ , where  $\|\mathbf{u}_i\|$  is the Euclidean norm on  $\mathbb{R}^n$ .

**Lemma 5.1.** *Under hypotheses (H1) and (H2), the following hold:*

- *For every displacement field  $\mathbf{u}$ ,  $\|M(\mathbf{u})\| \leq \|\mathbf{u}\|$ .*
- *If equality holds, then for any two points  $i \sim_G j$ , we have  $M(\mathbf{u})_i = \mathbf{u}_j$ .*

*Proof.* For each point  $i$ , we get by hypotheses (H1) and (H2) that  $\sum_{j \in \Lambda} \lambda_{ij} = \sum_{j \in \Lambda} \lambda_{ji} = 1$ . Then, for each direction  $\ell$ , Jensen's inequality [12] applied to the strictly convex function  $x \rightarrow x^2$  yields

$$(5.1) \quad \left[ \sum_{j \in \Lambda} \lambda_{ij} u_{j\ell} \right]^2 \leq \sum_{j \in \Lambda} \lambda_{ij} u_{j\ell}^2.$$

We can now write

$$(5.2) \quad \begin{aligned} \|M(\mathbf{u})\|^2 &= \sum_{\ell=1}^n \sum_{i \in \Lambda} \left[ \sum_{j \in \Lambda} \lambda_{ij} u_{j\ell} \right]^2 \\ &\leq \sum_{\ell=1}^n \sum_{i \in \Lambda} \sum_{j \in \Lambda} \lambda_{ij} u_{j\ell}^2 = \sum_{\ell=1}^n \sum_{j \in \Lambda} \sum_{i \in \Lambda} \lambda_{ij} u_{j\ell}^2 \\ &= \sum_{\ell=1}^n \sum_{j \in \Lambda} u_{j\ell}^2 \left[ \sum_{i \in \Lambda} \lambda_{ij} \right] = \sum_{\ell=1}^n \sum_{j \in \Lambda} u_{j\ell}^2 = \|\mathbf{u}\|^2. \end{aligned}$$

Moreover, let us suppose that  $\|M(\mathbf{u})\| = \|\mathbf{u}\|$ . Then, for each point  $i$  and each  $\ell$ , the equality in (5.1) is reached. Thus, the coordinates  $u_{j\ell}$  associated with the nonvanishing coefficients  $\lambda_{ij}$  are all identical (cf. [8]). This means that  $\mathbf{u}_j = \mathbf{u}_{j'}$  for any two points  $i \sim_G j$  and  $i \sim_G j'$ . Therefore, it follows that  $M(\mathbf{u})_i = \sum_{j' \in \Lambda} \lambda_{ij'} \mathbf{u}_{j'} = (\sum_{j' \in \Lambda} \lambda_{ij'}) \mathbf{u}_j = \mathbf{u}_j$ , where  $j$  is any point such that  $i \sim_G j$ . ■

**Lemma 5.2.** *For every displacement field  $\mathbf{u}$ , we have  $\|P(\mathbf{u})\| \leq \|\mathbf{u}\|$ . The equality holds if and only if  $\mathbf{u}_i$  is orthogonal to the gradient  $\nabla I_i$  at any point  $i$  of  $\Lambda$ . In that case,  $P(\mathbf{u})_i = \mathbf{u}_i$  at any point  $i$  of  $\Lambda$ .*

*Proof.* Let  $\mathbf{u}$  be a displacement field and  $i$  a point of the image. There exist a vector  $\vec{a}$  of  $\mathbb{R}^n$  and a real number  $b$  such that  $\nabla I_i^T \vec{a} = 0$  and  $\mathbf{u}_i = \vec{a} + b \nabla I_i$ . From the expression of  $P_i$ , we find  $P_i \vec{a} = \vec{a}$  (because  $\nabla I_i^T \vec{a} = 0$ ) and  $P_i \nabla I_i = (1 - \frac{\|\nabla I_i\|^2}{\alpha + \|\nabla I_i\|^2}) \nabla I_i$ . Thus,  $P_i \mathbf{u}_i = \vec{a} + b \nabla I_i (1 - \frac{\|\nabla I_i\|^2}{\alpha + \|\nabla I_i\|^2})$ . We notice here that  $\|\mathbf{u}_i\|^2 = \|\vec{a}\|^2 + b^2 \|\nabla I_i\|^2$  and  $\|P_i \mathbf{u}_i\|^2 = \|\vec{a}\|^2 + (1 - \frac{\|\nabla I_i\|^2}{\alpha + \|\nabla I_i\|^2})^2 b^2 \|\nabla I_i\|^2$ . So, we get that  $\|P_i \mathbf{u}_i\| \leq \|\mathbf{u}_i\|$ , and that the equality holds if and only if  $b = 0$  or  $\nabla I_i = \vec{0}$  (i.e.,  $\mathbf{u}_i = \vec{a}$ ), which means if and only if  $\nabla I_i^T \mathbf{u}_i = 0$ . Finally,  $\|P(\mathbf{u})\|^2 = \sum_{i \in \Lambda} \|P_i \mathbf{u}_i\|^2 \leq \sum_{i \in \Lambda} \|\mathbf{u}_i\|^2 = \|\mathbf{u}\|^2$ , with equality if and only if  $\nabla I_i^T \mathbf{u}_i = 0$  at every point  $i$  of  $\Lambda$ . In that case, it is clear that  $P_i \mathbf{u}_i = \mathbf{u}_i$  at every point  $i$  of  $\Lambda$ . ■

Let us recall that the HS iterations read as  $\mathbf{u}^{k+1} = PM(\mathbf{u}^k) + \mathbf{d}$ . We can now show the convergence of these iterations, under our condition on the intensity field. From Lemmas 5.1 and 5.2, we find that  $\|PM(\mathbf{u})\| \leq \|\mathbf{u}\|$  for any displacement field  $\mathbf{u}$ . A feature of the following proof consists in showing that  $\|(PM)^N(\mathbf{u})\| < \|\mathbf{u}\|$  for any nonzero displacement field  $\mathbf{u}$ , where  $N$  is the number of points in  $\Lambda$ .

*Proof of Theorem 4.1.* We still suppose that (H1), (H2), and (H3) are verified. Let us assume that the rank of  $(\nabla I_i)$  is  $n$ . Let us assume, by contradiction, that there is a displacement field  $\mathbf{u} \neq \mathbf{0}$  such that  $\|(PM)^N(\mathbf{u})\| = \|\mathbf{u}\|$ . So, there is a point  $i_* \in \Lambda$  such that  $\mathbf{u}_{i_*} \neq \vec{0}$ . Let  $i$  be any point of  $\Lambda$ . We claim that there is a path from  $i$  to  $i_*$  in the graph  $G$  of length  $1 \leq L \leq N$  ( $N$  is the number of elements in  $G$ ). Indeed, if  $i \neq i_*$ , then a minimal path will do; if  $i = i_*$ , then the path  $i_0 = i \sim_G i_1 \sim_G i_2 \sim_G \dots \sim i_L = i_*$  will do, where  $i_1$  is any neighbor of  $i$  in the graph  $G$ . Let this path be of the form  $i_0 = i \sim_G i_1 \sim_G i_2 \sim_G \dots \sim i_L = i_*$ . From Lemmas 5.1 and 5.2, the assumption  $\|(PM)^N(\mathbf{u})\| = \|\mathbf{u}\|$  implies that  $\|(PM)^L(\mathbf{u})\| = \|M(PM)^{L-1}(\mathbf{u})\| = \|(PM)^{L-1}(\mathbf{u})\| = \dots = \|\mathbf{u}\|$ . Moreover, again from Lemmas 5.1 and 5.2, we have  $(PM)^L(\mathbf{u})_{i_0} = M(PM)^{L-1}(\mathbf{u})_{i_0} = (PM)^{L-1}(\mathbf{u})_{i_1} = \dots = \mathbf{u}_{i_L}$ . Also, from Lemma 5.2, we have that  $M(PM)^{L-1}(\mathbf{u})_{i_0}$  is orthogonal to  $\nabla I_{i_0}$  and thus that  $\mathbf{u}_{i_L} = \mathbf{u}_{i_*}$  is orthogonal to  $\nabla I_{i_0} = \nabla I_i$ . Since the point  $i$  is arbitrary, we deduce that the space spanned by the gradient vectors  $\nabla I_i$  is orthogonal to the nonzero vector  $\mathbf{u}_{i_*}$ , which is a contradiction. Thus, under the condition of convergence stated in the theorem and for  $\mathbf{u} \neq \mathbf{0}$ , we have  $\|(PM)^N(\mathbf{u})\| < \|\mathbf{u}\|$ .

We now consider the function  $\mathbf{u} \rightarrow \|(PM)^N(\mathbf{u})\|$  defined on the hypersphere  $\{\mathbf{u} \mid \|\mathbf{u}\| = 1\}$ . This function is continuous and defined on a compact set, i.e., a bounded closed subset of the vector space of displacement fields. Therefore, the function is bounded and reaches its maximal value. This ensures that there exists  $\beta < 1$  such that for every displacement field  $\mathbf{u}$ ,  $\|(PM)^N(\mathbf{u})\| \leq \beta \|\mathbf{u}\|$ . Since, moreover,  $\|PM(\mathbf{u})\| \leq \|\mathbf{u}\|$ , the conclusion about the existence of a solution for the linear system (2.5), its uniqueness, and the convergence of the iterations (2.9) to this solution is then a classical result (see [19, p. 101], for example).

We now suppose that the rank of  $(\nabla I_i)$  is less than  $n$ . In this case, the intensity gradients are all contained in the same hyperplane. Let us consider a displacement field  $\mathbf{u}^*$  that is uniform, different from zero, and orthogonal to this hyperplane. Because of hypothesis (H2), which imposes  $\sum_{j \in \Lambda} \lambda_{ij} = 1$  at each point  $i$ , and because  $\mathbf{u}^*$  is uniform, we get  $M(\mathbf{u}^*) = \mathbf{u}^*$ . Moreover, because  $\nabla I_i^T \mathbf{u}_i^* = 0$  at each point  $i$ , Lemma 5.2 says that  $P(\mathbf{u}^*) = \mathbf{u}^*$ . Thus,  $PM(\mathbf{u}^*) = \mathbf{u}^*$ . This shows that the linear system  $\mathbf{u} = PM(\mathbf{u}) + \mathbf{d}$  (equivalent to the linear system (2.5)) has a nonzero solution when  $\mathbf{d} = \mathbf{0}$ , so that the coefficient matrix of the linear system (2.5) is not invertible. ■

## 6. The discrete Laplacian in dimension $n$ .

**6.1. Description of a general scheme.** Recall that the lattice  $\Lambda$  is assumed to be of the form  $\{(i_1, i_2, \dots, i_n) : i_\ell \text{ is an integer ranging from } 0 \text{ to } N_\ell - 1 \text{ for } 1 \leq \ell \leq n\}$ , where  $N_\ell \geq 1$  for  $1 \leq \ell \leq n$ . The lattice  $\Lambda$  is viewed as a subset of the Cartesian product  $\mathbb{Z}^\ell$ . In the following, the norms  $L^1$  and  $L^\infty$  are denoted by  $\|(i_1, i_2, \dots, i_n)\|_{L^1} = \sum_{\ell=1}^n |i_\ell|$  and  $\|(i_1, i_2, \dots, i_n)\|_{L^\infty} = \max_{1 \leq \ell \leq n} |i_\ell|$ . We will now define a general way of calculating a discrete Laplacian in dimension  $n$ , based on [15]. As proposed in [15], we consider the  $n$ -dimensional finite-difference stencil  $S_i$  around a point  $i$ , consisting of the  $3^n - 1$  points  $k \in \mathbb{Z}^\ell$  that verify  $\|i - k\|_{L^\infty} = 1$ . Then, we divide these stencil points into the sets  $S_i^{(r)}$  ( $1 \leq r \leq n$ ) of points  $k \in \mathbb{Z}^\ell$  that verify  $\|i - k\|_{L^1} = r$ . As explained in [15], it turns out that for each  $r$  in  $\{1, \dots, n\}$ , a discretization of the Laplacian can be constructed from the Taylor expansions of the points of  $S_i^{(r)}$  about the point  $i$ . The remaining part of this section concerns only interior points of the lattice  $\Lambda$ ; the boundary cases are discussed in section 6.2. So, if  $i$  is not a boundary point, the discretization of the Laplacian is given in [15, formula (2.2)]:

$$(6.1) \quad \Delta^{(r)}(\mathbf{u})_i = \kappa_r \sum_{k \in S_i^{(r)}} (\mathbf{u}_k - \mathbf{u}_i),$$

where  $\kappa_r = \frac{2n}{r \text{Card}(S_i^{(r)})}$ . Based on the definition of  $S_i^{(r)}$ , it is clear that  $\text{Card}(S_i^{(r)}) = \binom{n}{r} 2^r$ , where  $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ . Thus,  $\kappa_r$  is independent of the point  $i$ . Then, a general way to calculate a global discrete Laplacian at the point  $i$  is to make a weighted average of the Laplacians obtained for the different sets  $S_i^{(r)}$ . Such a discretization can be written as

$$(6.2) \quad \Delta(\mathbf{u})_i = \sum_{r=1}^n w_r \Delta^{(r)}(\mathbf{u})_i,$$

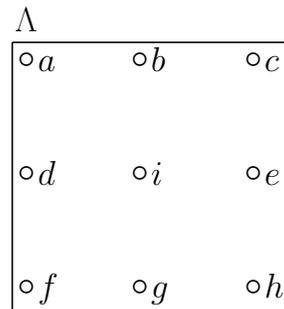
where the weights  $w_r \geq 0$  are nonnegative real numbers such that  $\sum_{r=1}^n w_r = 1$ . We also denote  $\kappa = \sum_{r=1}^n w_r \kappa_r \text{Card}(S_i^{(r)})$ , independent of  $i$ , and  $\gamma_r = \frac{w_r \kappa_r}{\kappa}$ , so that

$$(6.3) \quad \Delta(\mathbf{u})_i = \kappa \left\{ \left( \sum_{r=1}^n \sum_{k \in S_i^{(r)}} \gamma_r \mathbf{u}_k \right) - \mathbf{u}_i \right\}.$$

We notice here that the coefficients  $\gamma_r$  are nonnegative and verify

$$(6.4) \quad \sum_{r=1}^n \sum_{k \in S_i^{(r)}} \gamma_r = \sum_{r=1}^n \gamma_r \text{Card}(S_i^{(r)}) = \sum_{r=1}^n \frac{w_r \kappa_r}{\kappa} \text{Card}(S_i^{(r)}) = 1.$$

In the following, we will impose  $w_1 \neq 0$ . This is a natural hypothesis because it means that  $\Delta^{(1)}(\mathbf{u})_i$ , which is calculated from the closest neighbors of  $i$ , is taken into account in the Laplacian calculation at  $i$ . Therefore, we have  $\gamma_1 \neq 0$ . Note that a simple way of calculating a discrete Laplacian in dimension  $n$  is to set  $w_1 = 1$ . The *dimension independent Laplacian* given in [15] is obtained by setting  $w_r = \binom{n-1}{r-1} 2^{1-n}$ . These coefficients are chosen so that some properties of the smooth Laplacian are kept with the discrete Laplacian (see [15] for more details). The scheme chosen by Horn and Schunck [10] in the 2-dimensional case, detailed below, is obtained by setting  $w_1 = w_2 = \frac{1}{2}$  (which is the dimension independent Laplacian in the case  $n = 2$ ).



**Figure 1.** The 2-dimensional finite-difference stencil for the Laplacian calculation at an interior point  $i$  of a lattice  $\Lambda$ .

In Figure 1, as an example, the stencil  $S_i$  is composed of the points  $\{a, b, c, d, e, f, g, h\}$ . We have  $S_i^{(1)} = \{b, d, e, g\}$  and  $S_i^{(2)} = \{a, c, f, h\}$ . The scheme of Horn and Schunck [10] is  $\Delta(\mathbf{u})_i = \kappa \left\{ \frac{1}{6} (\mathbf{u}_b + \mathbf{u}_d + \mathbf{u}_e + \mathbf{u}_g) + \frac{1}{12} (\mathbf{u}_a + \mathbf{u}_c + \mathbf{u}_f + \mathbf{u}_h) - \mathbf{u}_i \right\}$ , with  $\kappa = 3$ . Here, the coefficient  $\gamma_1$  associated with  $S_i^{(1)}$  is  $\frac{1}{6}$ , and the coefficient  $\gamma_2$  associated with  $S_i^{(2)}$  is  $\frac{1}{12}$ .

Finally, from (2.4), the boundary conditions considered here are that the normal derivatives vanish at the boundary of the image. In [10], Horn and Schunck explained how to deal with these conditions: when a point outside the image is needed, the displacement of the closest point inside the image is copied.

The description of the discretization scheme given above is sufficient for programming the HS algorithm: just choose some coefficients  $w_r$ , calculate the corresponding coefficients  $\gamma_r$ , use (2.9) with  $M(\mathbf{u})_i = \sum_{r=1}^n \sum_{k \in S_i^{(r)}} \gamma_r \mathbf{u}_k$  (cf. (2.6) and (6.3)), and apply the boundary conditions when necessary.

**6.2. Determination of the weights in the average calculation.** We will now give an expression of the coefficients  $\lambda_{ij}$  defined in (2.7), in order to verify hypotheses (H1), (H2), and (H3) in the next section.

Let us give a rigorous definition of the boundary conditions. We denote  $\Lambda' = \{(k_1, k_2, \dots, k_n) : -1 \leq k_\ell \leq N_\ell, 1 \leq \ell \leq n\}$  and define the function  $f$  from  $\Lambda'$  to  $\Lambda$  such that  $f\{(k_1, k_2, \dots, k_n)\} = (j_1, j_2, \dots, j_n)$ , where, for each  $\ell$  in  $\{1, \dots, n\}$ , the following hold:

- If  $0 \leq k_\ell \leq N_\ell - 1$ , then  $j_\ell = k_\ell$ .
- If  $k_\ell = -1$ , then  $j_\ell = 0$ .
- If  $k_\ell = N_\ell$ , then  $j_\ell = N_\ell - 1$ .

Then, our discretization scheme can be written at each point  $i$  of  $\Lambda$ , even if  $i$  is a boundary point:

$$(6.5) \quad \Delta(\mathbf{u})_i = \kappa \left\{ \left( \sum_{r=1}^n \sum_{k \in S_i^{(r)}} \gamma_r \mathbf{u}_{f(k)} \right) - \mathbf{u}_i \right\}.$$

Now, given two points  $i$  and  $j$  of  $\Lambda$  and an integer  $r$  in  $\{1, \dots, n\}$ , we denote  $A_{ij}^{(r)}$  the set of points defined by

$$(6.6) \quad A_{ij}^{(r)} = \{k \in S_i^{(r)} \subset \Lambda' : f(k) = j\}.$$

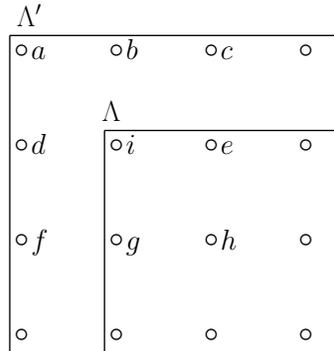
Then, for two points  $i$  and  $j$  of  $\Lambda$ , we set

$$(6.7) \quad \lambda_{ij} = \sum_{r=1}^n \text{Card}(A_{ij}^{(r)}) \gamma_r.$$

It is clear that at each point  $i$  of  $\Lambda$  and for every displacement field  $u$

$$(6.8) \quad \sum_{j \in \Lambda} \lambda_{ij} \mathbf{u}_j = \sum_{r=1}^n \sum_{k \in S_i^{(r)}} \gamma_r \mathbf{u}_{f(k)}.$$

Thus, from (6.5), our discretization scheme can be written as in (2.6) and (2.7):  $\Delta(\mathbf{u})_i = \kappa \{M(\mathbf{u})_i - \mathbf{u}_i\}$ , with  $M(\mathbf{u})_i = \sum_{j \in \Lambda} \lambda_{ij} \mathbf{u}_j$ .



**Figure 2.** The 2-dimensional finite-difference stencil for the Laplacian calculation at a boundary (corner) point  $i$ .

In Figure 2, as a complement to the example of Figure 1, the stencil  $S_i$  of the boundary point  $i$  (located at a corner of the lattice  $\Lambda$ ) is composed of the points  $\{a, b, c, d, e, f, g, h\}$ . As for Figure 1, we have  $S_i^{(1)} = \{b, d, e, g\}$  and  $S_i^{(2)} = \{a, c, f, h\}$ . Based on the definition of (6.6), one obtains  $A_{i,i}^{(1)} = \{b, d\}$  and  $A_{i,i}^{(2)} = \{a\}$ ;  $A_{i,e}^{(1)} = \{e\}$  and  $A_{i,e}^{(2)} = \{c\}$ ;  $A_{i,g}^{(1)} = \{g\}$  and  $A_{i,g}^{(2)} = \{f\}$ ;  $A_{i,h}^{(1)} = \emptyset$  and  $A_{i,h}^{(2)} = \{h\}$ . The corresponding scheme of Horn and Schunck [10] is  $\Delta(\mathbf{u})_i = \kappa \left\{ \frac{1}{6} (2\mathbf{u}_i + \mathbf{u}_e + \mathbf{u}_g) + \frac{1}{12} (\mathbf{u}_i + \mathbf{u}_e + \mathbf{u}_g + \mathbf{u}_h) - \mathbf{u}_i \right\} = \kappa \left\{ \frac{1}{12} (5\mathbf{u}_i + 3\mathbf{u}_e + 3\mathbf{u}_g + \mathbf{u}_h) - \mathbf{u}_i \right\}$ , with  $\kappa = 3$ . Again, the coefficient  $\gamma_1$  associated with  $S_i^{(1)}$  is  $\frac{1}{6}$ , and the coefficient  $\gamma_2$  associated with  $S_i^{(2)}$  is  $\frac{1}{12}$ . The case of a boundary point not located at the corner of  $\Lambda$  (such as the point  $e$  in Figure 2) can be treated similarly.

**7. Verification of the hypotheses.** We now have to verify that the general  $n$ -dimensional scheme described in section 6 fulfills the hypotheses of section 4:

- (H1) For all points  $i$  and  $j$  of  $\Lambda$ ,  $\lambda_{ij} = \lambda_{ji}$ .
- (H2) At every point  $i$  of  $\Lambda$ ,  $\sum_{j \in \Lambda} \lambda_{ij} = 1$ .
- (H3) The graph  $G$  is connected.

**Proposition 7.1.** *With the discretization scheme of section 6, (H1) is satisfied.*

*Proof.* Let  $i$  and  $j$  be two points of  $\Lambda$ , and let  $r$  be an integer in  $\{1, \dots, n\}$ . As in section 6.2, we denote  $A_{ij}^{(r)}$  the set of points  $k$  belonging to  $S_i^{(r)}$  and satisfying  $f(k) = j$ . By definition, a point  $k$  belongs to  $A_{ij}^{(r)}$  if and only if the following hold:

- $\|k - i\|_{L^\infty} = 1$ ;
- $\|k - i\|_{L^1} = r$ ;
- $f(k) = j$ .

Let us now define the function  $g_{ij}$  from  $\mathbb{N}^n$  to  $\mathbb{N}^n$  by  $g_{ij}(k) = k + i - j$ . We will show that  $g_{ij}(A_{ij}^{(r)}) \subset A_{ji}^{(r)}$ . We denote  $i = (i_1, i_2, \dots, i_n)$  and  $j = (j_1, j_2, \dots, j_n)$ . Let  $k = (k_1, k_2, \dots, k_n)$  be a point of  $A_{ij}^{(r)}$ , and let  $\ell$  be an integer in  $\{1, \dots, n\}$ .

- If  $k_\ell = -1$ , then  $j_\ell = 0$  (because  $f(k) = j$ ) and  $i_\ell = 0$  (because  $\|k - i\|_{L^\infty} \leq 1$  and  $i$  belongs to  $\Lambda$ ), so that  $k_\ell + i_\ell - j_\ell = -1$ .
- If  $k_\ell = N_\ell$ , similarly,  $j_\ell = i_\ell = N_\ell - 1$  and  $k_\ell + i_\ell - j_\ell = N_\ell$ .
- In the other cases,  $k_\ell = j_\ell$  (because  $f(k) = j$ ), so that  $k_\ell + i_\ell - j_\ell = i_\ell$ .

First, from the definition of the function  $f$ , the three previous cases applied to each coordinate  $l$  of  $\{1, \dots, n\}$  yield  $f(g_{ij}(k)) = f(k + i - j) = i$ . Moreover, in each of these three cases, it is clear that  $|(k_\ell + i_\ell - j_\ell) - j_\ell| = |k_\ell - i_\ell|$  (either because  $j_\ell = i_\ell$  or because  $k_\ell = j_\ell$ ). Thus, from  $\|k - i\|_{L^\infty} \leq 1$  and  $\|k - i\|_{L^1} = r$ , we obtain  $\|g_{ij}(k) - j\|_{L^\infty} = 1$  and  $\|g_{ij}(k) - j\|_{L^1} = r$ . This permits us to conclude that  $g_{ij}(A_{ij}^{(r)}) \subset A_{ji}^{(r)}$ .

Now, as  $g_{ij}$  is a translation, it is injective, so that  $\text{Card}(A_{ij}^{(r)}) \leq \text{Card}(A_{ji}^{(r)})$ . Then, as we did not impose any hypothesis about  $i$  and  $j$ , we can exchange them and write  $\text{Card}(A_{ji}^{(r)}) \leq \text{Card}(A_{ij}^{(r)})$ , so that  $\text{Card}(A_{ij}^{(r)}) = \text{Card}(A_{ji}^{(r)})$ . Finally, (6.7) imposes that  $\lambda_{ij} = \lambda_{ji}$ , so that (H1) is verified. ■

**Proposition 7.2.** *With the discretization scheme of section 6, (H2) is satisfied.*

*Proof.* Let  $i$  be a point of  $\Lambda$ . Equation (6.8) applied to a displacement field  $u$  that is uniform and different from zero yields  $\sum_{r=1}^n \sum_{k \in S_i^{(r)}} \gamma_r = \sum_{j \in \Lambda} \lambda_{ij}$ . Then, (6.4) imposes  $\sum_{j \in \Lambda} \lambda_{ij} = 1$ , so that (H2) is verified. ■

**Proposition 7.3.** *With the discretization scheme of section 6, (H3) is satisfied.*

*Proof.* Let  $i$  be a point of the image lattice  $\Lambda$ . By hypothesis, we have that  $\gamma_1 \neq 0$ . So, from (6.7), we have that for two close neighbors  $i$  and  $j$  of  $\Lambda$ , i.e., such that  $\|i - j\|_{L^1} = 1$ ,  $\lambda_{ij} \neq 0$ . Indeed, in that case,  $j$  belongs to  $A_{ij}^{(1)}$ , so that  $\text{Card}(A_{ij}^{(1)}) \neq 0$ . So, two close neighbors are always linked in  $G$ , and the connectedness of  $G$  becomes obvious. ■

So, (H1), (H2), and (H3) are fulfilled with the discretization scheme of section 6, and we are under the conditions of Theorem 4.1.

**8. Conclusion.** The proposed convergence result was shown using a general definition of the discrete Laplacian. That definition includes the classical scheme of Horn and Schunck

in dimension 2 and a general scheme (see section 6) for  $n$ -dimensional Laplacians. In this context, a necessary and sufficient condition for the problem to be well-posed (i.e., to have a unique solution) is that the intensity gradients not all be contained in the same hyperplane. Under that condition, the HS iterations converge to the solution. It was also shown that the convergence of the HS iterative scheme implies the convergence of the Gauss–Seidel and SOR solvers for the HS problem.

**Appendix A.** Here the details of the derivation of (2.9) in dimension  $n \geq 1$  are presented. From (2.6) and (2.8), equation (2.5) is equivalent to

$$(A.1) \quad (\alpha \mathcal{I}_n + [\nabla I \nabla I^T]_i) \mathbf{u}_i - \alpha M(\mathbf{u})_i = -I_{t,i} \nabla I_i,$$

where  $\mathcal{I}_n$  denotes the  $n \times n$  identity matrix. Let us now notice that

$$(A.2) \quad [\nabla I \nabla I^T]_i^2 = \nabla I_i [\nabla I^T \nabla I]_i \nabla I_i^T = \|\nabla I_i\|^2 [\nabla I \nabla I^T]_i.$$

Thus,

$$(A.3) \quad \begin{aligned} & (\alpha \mathcal{I}_n + [\nabla I \nabla I^T]_i) (\alpha \mathcal{I}_n + \|\nabla I_i\|^2 \mathcal{I}_n - [\nabla I \nabla I^T]_i) \\ & = \alpha (\alpha + \|\nabla I_i\|^2) \mathcal{I}_n. \end{aligned}$$

So,

$$(A.4) \quad \begin{aligned} (\alpha \mathcal{I}_n + [\nabla I \nabla I^T]_i)^{-1} &= \frac{\alpha \mathcal{I}_n + \|\nabla I_i\|^2 \mathcal{I}_n - [\nabla I \nabla I^T]_i}{\alpha (\alpha + \|\nabla I_i\|^2)} \\ &= \alpha^{-1} \mathcal{I}_n - \alpha^{-1} \frac{[\nabla I \nabla I^T]_i}{\alpha + \|\nabla I_i\|^2}. \end{aligned}$$

We also have

$$(A.5) \quad [\nabla I \nabla I^T]_i I_{t,i} \nabla I_i = I_{t,i} \nabla I_i [\nabla I^T \nabla I]_i = \|\nabla I_i\|^2 I_{t,i} \nabla I_i.$$

Now, from (A.4) and (A.5), the expression (A.1) can be rewritten as

$$(A.6) \quad \mathbf{u}_i = \left( \mathcal{I}_n - \frac{[\nabla I \nabla I^T]_i}{\alpha + \|\nabla I_i\|^2} \right) M(\mathbf{u})_i - \frac{I_{t,i} \nabla I_i}{\alpha + \|\nabla I_i\|^2}.$$

This equality leads us to write the general HS iterations for an  $n$ -dimensional image:

$$(A.7) \quad \mathbf{u}_i^{k+1} = \left( \mathcal{I}_n - \frac{[\nabla I \nabla I^T]_i}{\alpha + \|\nabla I_i\|^2} \right) M(\mathbf{u}^k)_i - \frac{I_{t,i} \nabla I_i}{\alpha + \|\nabla I_i\|^2}.$$

With the notation introduced in section 2, we thus obtain (2.9).

**Appendix B.** We discuss the condition of block diagonally dominant matrices in the context of the Jacobi solver for the HS problem. We refer the reader to [11, 1] for results on the convergence of the Jacobi method for strictly diagonally dominant matrices or irreducible and weakly diagonally dominant matrices, as well as [7] for the corresponding notions in the case of block matrices.

First, using (2.5), (2.6), (2.7), and (2.8), we observe that (2.5) can be rewritten in the form (see (A.1))

$$(B.1) \quad \{\alpha \mathbf{u}_i + [\nabla I \nabla I^T]_i \mathbf{u}_i\} - \sum_{j=1}^N \alpha \lambda_{ij} \mathbf{u}_j = -I_{t,i} \nabla I_i.$$

Let  $A_{ij}$ , for  $i, j \in \Lambda$ , be the  $n \times n$  matrices defined by

$$(B.2) \quad A_{ij} = -\alpha \lambda_{ij} \mathcal{I}_n, \quad i \neq j;$$

$$(B.3) \quad A_{ii} = (\alpha \mathcal{I}_n + [\nabla I \nabla I^T]_i) - \alpha \lambda_{ii} \mathcal{I}_n.$$

Then, the Jacobi iteration is expressed as

$$(B.4) \quad \mathbf{u}_i^{k+1} = A_{ii}^{-1} \left( - \sum_{j \neq i} A_{ij} \mathbf{u}_j^k - I_{t,i} \nabla I_i \right).$$

**Lemma B.1.** *Assume that  $0 \leq \lambda_{ii} < 1$ . Then, the inverse matrix of the block  $A_{ii}$  is equal to  $\frac{1}{\alpha(1-\lambda_{ii})} P'_i$ , where  $P'_i = \mathcal{I}_n - \frac{[\nabla I \nabla I^T]_i}{\alpha(1-\lambda_{ii}) + \|\nabla I_i\|^2}$ .*

*Proof.* The lemma follows directly from (A.4) upon replacing  $\alpha$  by  $\alpha' = \alpha(1 - \lambda_{ii})$ . ■

So, let  $i$  be a point in the interior of  $\Lambda$ . From section 6,  $\lambda_{ii}$  is then equal to 0. Then,  $P'_i = P_i$  and the Jacobi iteration for the point  $i$  of (B.4) reads as

$$(B.5) \quad \mathbf{u}_i^{k+1} = \alpha^{-1} P_i \left( \sum_{j \neq i} \alpha \lambda_{ij} \mathbf{u}_j^k - I_{t,i} \nabla I_i \right)$$

$$(B.6) \quad = P_i M(\mathbf{u}^k)_i + \mathbf{d}_i,$$

which amounts to the HS iteration (2.9). On the other hand, since  $\lambda_{ii} \neq 0$  if  $i$  is a boundary point, the Jacobi iteration is never the HS iteration at boundary points.

Let  $\|P'_i\|$  be the norm of the matrix  $P'_i$  defined by  $\max_{\mathbf{u}_i \neq 0} \frac{\|P'_i \mathbf{u}_i\|}{\|\mathbf{u}_i\|}$  based on any norm of  $\mathbb{R}^n$ .

**Lemma B.2.** *Let  $n \geq 2$ , and consider a vector  $\mathbf{u}_i$  in  $\mathbb{R}^n$  that is orthogonal to  $\nabla I_i$ . Then,  $P'_i(\mathbf{u}_i) = \mathbf{u}_i$ . Therefore,  $\|P'_i\| \geq 1$  no matter the norm used on  $\mathbb{R}^n$ .*

*Proof.* This result follows directly from the proof of Lemma 5.2. ■

**Lemma B.3.** *If  $n \geq 2$  and hypothesis (H2) is fulfilled, then  $\|A_{ii}^{-1}\|^{-1} \leq \sum_{j \neq i} \|A_{ij}\|$ , for any  $i$ , no matter the norm used on  $\mathbb{R}^n$ .*

*Proof.* From the definition (B.2) of  $A_{ij}$ ,  $j \neq i$ , we have  $\sum_{j \neq i} \|A_{ij}\| = \alpha \sum_{j \neq i} \lambda_{ij}$ . Then, from Lemmas B.1 and B.2 and hypothesis (H2), we have  $\|A_{ii}^{-1}\|^{-1} = \alpha(1 - \lambda_{ii}) \|P'_i\|^{-1} \leq \alpha(1 - \lambda_{ii}) = \alpha \sum_{j \neq i} \lambda_{ij}$ . ■

From Lemma B.3, one concludes that the matrix  $A$  is never weakly (or strictly) block diagonally dominant if  $n \geq 2$  under hypothesis (H2).<sup>1</sup> On the other hand, if one uses the Euclidean norm on  $\mathbb{R}^n$ , one can easily show that  $\|A_{ii}^{-1}\|^{-1} = \sum_{j \neq i} \|A_{ij}\|$ , for any  $i$ , because

<sup>1</sup>Recall that a matrix  $A$  is weakly (or strictly) block diagonally dominant if  $\|A_{ii}^{-1}\|^{-1} \geq \sum_{j \neq i} \|A_{ij}\|$  for any  $i$  and if that inequality is strict for some (or any, respectively)  $i$ .

$\|P'_i\| = 1$  for that norm. So, in that case,  $A$  is block diagonally dominant (i.e.,  $\|A_{ii}^{-1}\|^{-1} \geq \sum_{j \neq i} \|A_{ij}\|$  for any  $i$ ), but the inequality is never strict.

Next, we show that the matrix  $A$  defined by (B.2) and (B.3) is not diagonally dominant if  $n \geq 2$  (here, the matrix is not viewed as a block matrix), except in very special cases. We first treat the case  $n = 2$ . The absolute values of the diagonal elements of the matrix  $A_{ii}$  are equal to  $\alpha(1 - \lambda_{ii}) + I_{x,i}^2$  and  $\alpha(1 - \lambda_{ii}) + I_{y,i}^2$ , whereas the sum of the absolute values of the elements off the diagonal for the corresponding rows of the matrix  $A$  are equal to  $\sum_{j \neq i} \alpha \lambda_{ij} + |I_{x,i}| |I_{y,i}|$  and  $\sum_{j \neq i} \alpha \lambda_{ij} + |I_{y,i}| |I_{x,i}|$ . Using the identity  $\sum_j \lambda_{ij} = 1$ , diagonal dominance is then equivalent to  $I_{x,i}^2 \geq |I_{x,i}| |I_{y,i}|$  and  $I_{y,i}^2 \geq |I_{y,i}| |I_{x,i}|$ , which implies that  $|I_{x,i}| = |I_{y,i}|$  for each  $i$  such that  $I_{x,i}$  and  $I_{y,i}$  are both different from 0. This is a very special case, so that the assertion that the matrix  $A$  is diagonally dominant (in general) is false. If  $n > 2$ , the absolute values of the diagonal elements of the matrix  $A_{ii}$  are equal to  $\alpha(1 - \lambda_{ii}) + I_{x_\ell,i}^2$  for  $1 \leq \ell \leq n$ , whereas the sum of the absolute values of the elements off the diagonal for the corresponding rows of the matrix  $A$  are equal to  $\sum_{j \neq i} \alpha \lambda_{ij} + |I_{x_\ell,i}| \sum_{\ell' \neq \ell} |I_{x_{\ell'},i}|$  for  $1 \leq \ell \leq n$ . Therefore, the diagonal dominance of  $A$  implies that  $\sum_{\ell=1}^n |I_{x_\ell,i}| \geq (n-1) \sum_{\ell=1}^n |I_{x_\ell,i}|$ , which implies that  $\nabla I_i = \vec{0}$  for each point  $i$ . So, again, the assertion that the matrix  $A$  is diagonally dominant (in general) is false. Therefore, it appears that the short argument given in [23, p. 249] for the convergence of the pointwise Jacobi method is erroneous.

#### Remarks.

1. The HS iterative scheme amounts to the Jacobi iterative scheme at the interior points of the image, but never at its boundary points. But then we believe that it is usually the HS scheme that is implemented rather than the Jacobi method. Indeed, it is easy to implement (cf. the end of section 6.1), still fully parallelizable, and it is the original method proposed by Horn and Schunck. The difference between the two schemes is due to the Neumann boundary conditions (because then  $\lambda_{ii} \neq 0$  at a boundary point).
2. The Neumann boundary conditions (2.4) that come from the unconstrained minimization problem are very important. In particular, they imply that the Laplacian of a uniform displacement field vanishes, i.e.,  $\Delta(\mathbf{u})_i = \kappa(M(\mathbf{u})_i - \mathbf{u}_i) = \kappa(\sum_{j \in \Lambda} \lambda_{ij} - 1)\mathbf{u}_i$ , so that we must have  $\sum_{j \in \Lambda} \lambda_{ij} = 1$ .
3. Due to this condition, known convergence results of the (block) Jacobi and Gauss–Seidel methods do not apply, unless  $n = 1$ . The result [1, Theorem 1, (a)] assumes that the matrix  $A$  is strictly diagonally dominant, which is not the case here. Also, the result [1, Theorem 1, (b)] assumes that  $A$  is irreducible and weakly diagonally dominant, which is not the case either. Note that one can generalize [1, Theorem 1] using the notion of block diagonally dominant matrices [7]; namely, one can prove along the lines of [1] that if  $A$  is strictly block diagonally dominant or if it is block irreducible and weakly diagonally dominant, then both the block Jacobi and the Gauss–Seidel solvers converge. But again, these hypotheses never hold for the HS problem, unless  $n = 1$ .
4. On the other hand, if one wants to relax the boundary condition (2.4) and allow  $\sum_{j \in \Lambda} \lambda_{ij} < 1$  at a boundary point, then one can show that  $A$  is weakly block diagonally dominant for the Euclidean norm and block irreducible (based on the connectedness of the graph  $G$ , i.e., hypothesis (H3)), so that both the block Jacobi and the Gauss–Seidel

solvers then converge. This may happen if one considers a minimization problem with constraints, for instance if the displacement is known at some points of the image.

**Appendix C.** In this appendix, we discuss the implications of Theorem 4.1 (i.e., the convergence of the HS method) on the convergence of the Gauss–Seidel and SOR iterative schemes through the property of positive definiteness of the coefficient matrix of the HS problem. We also present a more general result that states conditions under which the convergence of the Gauss–Seidel and SOR methods is implied by the convergence of the Jacobi method. In what follows,  $\rho(A)$  denotes the spectral radius of a square matrix  $A$ .

**Proposition C.1.** *Let  $\tilde{B}$  and  $\tilde{C}$  be real symmetric matrices of the same dimensions such that  $\tilde{B}$  is positive definite and  $\rho(\tilde{B}^{-1}\tilde{C}) < 1$ . Then, the matrix  $\tilde{B} + \tilde{C}$  is symmetric positive definite.*

*Proof.* Since the matrix  $\tilde{B}$  is symmetric positive definite, it can be expressed in the form  $LL$ , where  $L$  is a symmetric invertible matrix. Indeed, one can write  $\tilde{B} = R\Psi R^T$ , where  $RR^T = \mathcal{I}$  ( $\mathcal{I}$  is the identity matrix) and  $\Psi$  is a diagonal positive definite matrix; thus,  $\tilde{B} = LL$ , where  $L = R\Psi^{1/2}R^T$ . Then,  $A = \tilde{B} + \tilde{C} = L(\mathcal{I} + L^{-1}\tilde{C}L^{-1})L$ . Since  $L$  is symmetric and invertible, then  $A$  is positive definite if and only if the symmetric matrix  $A' = \mathcal{I} + L^{-1}\tilde{C}L^{-1}$  is positive definite. Now, one has that  $\rho(L^{-1}\tilde{C}L^{-1}) = \rho(L^{-1}L^{-1}\tilde{C}L^{-1}L) = \rho(\tilde{B}^{-1}\tilde{C}) < 1$ . Therefore, the real symmetric matrix  $L^{-1}\tilde{C}L^{-1}$  can be written as  $Q^T\Lambda Q$ , where  $Q^TQ = \mathcal{I}$  and  $\Lambda$  is a diagonal matrix such that  $\rho(\Lambda) = \rho(L^{-1}\tilde{C}L^{-1}) < 1$ . It follows that  $A' = Q^T(\mathcal{I} + \Lambda)Q$ , where  $\mathcal{I} + \Lambda$  is a diagonal positive definite matrix (because any eigenvalue  $\lambda$  of  $\Lambda$  is such that  $|\lambda| < 1$ ). Thus,  $A'$  is a symmetric positive definite matrix, and so is  $A$ . ■

**Corollary C.2.** *Let  $Ax = \mathbf{b}$  be a linear system, where  $A$  is a real symmetric matrix. Let  $A$  be written in the form  $D - B - C$ , where  $D$ ,  $B$ , and  $C$  are block diagonal, block upper triangular, and block lower triangular matrices, respectively. Assume that  $D$  is positive definite. Then, the convergence of the Jacobi iterative scheme  $\mathbf{x}^{k+1} = D^{-1}((B + C)\mathbf{x}^{k+1} + \mathbf{b})$  implies the convergence of the Gauss–Seidel and SOR iterative schemes. In fact, the matrix  $A$  is positive definite under the assumptions.*

*Proof.* Let  $\tilde{B} = D$  and  $\tilde{C} = -B - C$ . The convergence of the Jacobi iterative scheme is equivalent to  $\rho(D^{-1}(B + C)) < 1$ . Thus, from Proposition C.1, the matrix  $A$  is positive definite. Henceforth, the Gauss–Seidel and SOR methods converge; see, for instance, [4, Theorem 5.3-2]. ■

**Corollary C.3.** *Under hypotheses (H1), (H2), and (H3), assume that the rank of  $(\nabla I_i)$  is  $n$ . Then, the coefficient matrix  $A$  of (B.1), with blocks defined by (B.2) and (B.3), is symmetric positive definite. In particular, the Gauss–Seidel and SOR iterative schemes converge under these conditions.*

*Proof.* Let  $\tilde{B} = \alpha P^{-1}$  and  $\tilde{C} = -\alpha M$ , where  $P$  and  $M$  are as in (2.9). Then,  $\tilde{B}$  is the block diagonal matrix with diagonal matrix entries  $A'_{ii} = \alpha \mathcal{I}_n + [\nabla I \nabla I^T]_i$ , as follows from Appendix A. Moreover, the eigenvalues of  $A'_{ii}$  are  $\alpha$  with multiplicity  $n - 1$  and  $\alpha + \|\nabla I_i\|^2$  with multiplicity 1. Thus, the symmetric matrix  $\tilde{B}$  is positive definite. Also, Theorem 4.1 implies that  $\rho(\tilde{B}^{-1}\tilde{C}) = \rho(PM) < 1$ . Finally,  $A = \alpha P^{-1} - \alpha M = \tilde{B} + \tilde{C}$ , using Appendix A. The statement on the positive definiteness of the matrix  $A$  now follows from Proposition C.1 since  $M$  is symmetric. Hence, the Gauss–Seidel and SOR iterative schemes converge under these conditions, as in the proof of Corollary C.2. ■

*Remark.* The positive definiteness of the coefficient matrix of the HS problem has been proved directly in [17]. Moreover, as mentioned in section 1, the V-ellipticity of the HS functional [18] implies the positive definiteness of the coefficient matrix of the HS problem. Thus, Corollary C.3 is not a new result. However, the more general result, Corollary C.2, might be of interest to further understand the convergence of the Jacobi, Gauss–Seidel, and SOR methods.

**Appendix D.** In this appendix, we give more details to explain why we think the proofs presented in [17, 13] are erroneous. We show that the matrix “ $P$ ” of [17, eq. (9)] (denoted here by  $P_*$  to avoid confusion with the linear transformation  $P$  of (2.9)) is not contracting for the norm defined by [17, eq. (10)], for any nonuniform image. Indeed, let  $i_0$  be a point where  $\nabla I_{i_0}^T = (I_{x,i_0}, I_{y,i_0}) \neq (0, 0)$ . We consider the displacement field  $\mathbf{u}$  defined by  $u_{2i-1} = I_{y,i_0}$  and  $u_{2i} = -I_{x,i_0}$  if  $i \in N_{i_0}$  (the set of four neighbors of  $i_0$ ), and  $u_{2i-1} = u_{2i} = 0$  otherwise. The norm defined in [17, eq. (10)], denoted by  $\|\cdot\|_*$  here to avoid any confusion, reads as  $\|\mathbf{u}\|_* = \max_{1 \leq i \leq N} (u_{2i-1}^2 + u_{2i}^2)^{\frac{1}{2}}$ . In that case, we obtain  $\|\mathbf{u}\|_* = (I_{x,i_0}^2 + I_{y,i_0}^2)^{\frac{1}{2}}$ . Moreover, we find that  $P_*(\mathbf{u})_{2i_0-1} = I_{y,i_0}$  and  $P_*(\mathbf{u})_{2i_0} = -I_{x,i_0}$ . Therefore,  $\|P_*(\mathbf{u})\|_* \geq (I_{x,i_0}^2 + I_{y,i_0}^2)^{\frac{1}{2}}$ , so that  $\|P_*(\mathbf{u})\|_* \geq \|\mathbf{u}\|_*$ . Thus,  $P_*$  is not contracting, due to this counterexample. We think that the error occurred in [17, formula (13)]: a factor  $c_i$  should be added in the second member to take into account that the sum in the first term includes all the neighbors of  $i$ . Thus, in the inequality [17, formula (15)], one should use the factor  $\sqrt{2}$  instead of 1, which makes that proof break down.

In [13, eq. (20)], the Laplacian corresponding to the Neumann boundary conditions (which usually correspond to the HS problem) is denoted by  $L_2$ . The matrix  $N_2$  is defined by the relation  $N_2(\mathbf{u}) = L_2(\mathbf{u}) + \mathbf{u}$  (cf. [13, eq. (22)]).<sup>2</sup> Since that Laplacian operator vanishes on uniform displacement fields, any such displacement field is an eigenvector of the matrix  $N_2$  for the eigenvalue 1. Therefore, the assertion after [13, eq. (23)] that the spectral radius  $\rho(N_2)$  of the matrix  $N_2$  (i.e., the maximal modulus of the eigenvalues of  $N_2$ ) is less than 1 is erroneous. Incidentally, in [13, formula (22)], a factor  $\frac{1}{2}$  is missing to get a correct expression of the average. In [13, formulas (38) and (40)], the authors also assert that  $\rho(I_d - F^{-1}\text{Diag}(S_{ij})) < 1$ . But, at every point, the determinant of the  $2 \times 2$  matrix  $S_{ij}$  is null.<sup>3</sup> Then, the matrices  $S_{ij}$  are singular, and so is  $F^{-1}\text{Diag}(S_{ij})$ . Thus, 1 is an eigenvalue of  $I_d - F^{-1}\text{Diag}(S_{ij})$ , and so the assertion is flawed. Thus, the two main intermediate results of [13] are both erroneous.

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<sup>2</sup>In our notation,  $L_2 = \Delta^{(1)}$  of (6.1) and  $N_2 = M$  of (2.7).

<sup>3</sup>In our notation, “ $(i, j)$ ” corresponds to a point  $i$  and “ $S_{ij}$ ” corresponds to the  $n \times n$  matrix  $[\nabla I \nabla I^T]_i$ . “ $I_d$ ” corresponds to the  $n \times n$  identity matrix  $I_n$ .

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